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INVESTIGATION OF THE STABILITY OF PERFECT GAS FLOW IN A QUASI-CYLINDRICAL CHANNEL

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Stability of the deceleration flow of a perfect, i.e. inviscid and non-heat-conducting gas in a channel with transition through the speed of sound in a normal compression shock or in a discontinuity of a more general kind (e.g. in a detonation wave) is analyzed in the one-dimensional ("hydraulic") approximation. Investigation is based on the assumption of smallness of perturbations and quasi-cylindricity of the channel. This permits the use of the method developed by Chernyi in 1953, and later used for solving a number of problems of flow dynamics in channels [1, 2] and also, for analyzing the stability of a two-front detonation wave [3]. In problems of flow stability in channels, the stream at the inlet is considered to be supersonic and unperturbed. At the channel outlet the condition of reflection is specified in the form of a linear relationship which defines perturbations of the "left-hand" Riemann invariant in terms of perturbations of the "right-hand" Riemann invariant and entropy. Necessary and sufficient conditions are obtained for cases in which any of the reflection coefficients is zero. These conditions are then used for analyzing flow stability in a channel with a closing compression shock. Besides the mentioned above Chernyi's investigations of stability of flow in a channel with a closing compression shock without any perturbation reflection from the channel outlet and the similar investigations by Babitskii in 1959, several other authors had considered this problem [4–9]. Unfortunately the results of their investigations, as a rule, lack strictness in spite of some valid assumptions (such as Chernyi's assumption of the channel quasi-cylindricity). Investigations of flows with a closing shock at the channel outlet under stationary conditions [4, 8] and of the flow in a convergent channel with the shock fairly close to the outlet [8] represent exceptions from this point of view. Remaining investigations are based either on some quasi-stationary assumptions of a qualitative nature [5] or on the supplementary assumption of shock stability in a divergent channel, or take into consideration only linear terms of expansions in terms of the space variable in conditions when it is necessary to take into account subsequent terms of the series [8]. Some of these are simply false [9], owing to an incorrect linearization of relationships at the shock (*). In all of the investigations cited above, as well as in the present paper, the linear theory is used. Note that numerical methods are recently finding increasing application in the analysis of flow stability and other dynamic processes in channels (see, e.g. [10 – 14]).

*) V. N. Glaznev, the author of [9] was acquainted with the relevant considerations and agreed with these.

1. Let us consider the flow in a channel whose cross-sectional area F is a known function of the longitudinal coordinate x measured along the channel axis. We use the following notation: t for time, u for the x-component of the stream velocity, p for pressure, ρ for density, e for the specific internal energy, i for the specific enthalpy, and a for the speed of sound, with

$$e = e(p, \rho), \quad i = i(p, \rho) = e + p / \rho, \quad a = a(p, \rho) \tag{1.1}$$

where the functions appearing on the right are known. For a perfect gas with adiabatic exponent $\varkappa$

$$e = p / (\varkappa - 1)\rho, \quad i = \varkappa p / (\varkappa - 1)\rho, \quad a = (\varkappa p / \rho)^{1/2}$$

The above quantities and other parameters without the superscript "°" are considered to be dimensionless. If $l^\circ, F_*^\circ, u_*^\circ$ and ρ_*° denote characteristic quantities whose dimensions are, respectively, those of length, area, velocity and density, their dimensionless form is obtained by reducing the longitudinal coordinate with respect to l°, the channel cross sectional area to F_*°, time to l°/u_*°, velocity of gas and the speed of sound to u_*°, density to ρ_*°, pressure to $\rho^\circ_* u_*^{\circ 2}$, and the specific energy and enthalpy with repect to $u_*^{\circ 2}$. Note that in the one-dimensional approximation considered here the characteristic area F_*° can be conveniently chosen independently of l°, while in an exact formulation $F^\circ_* \sim l^{\circ 2}$.

In the considered approximation the equations which define the nonstationary flow of an inviscid and non-heat-conducting gas in a channel with impermeable walls reduces to the "characteristic" system

$$\begin{gathered}\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} - a^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) = 0 \\ \frac{\partial u}{\partial t} + (u + a)\frac{\partial u}{\partial x} + \frac{1}{\rho a}\left\{\frac{\partial p}{\partial t} + (u + a)\frac{\partial p}{\partial x}\right\} + ua\frac{d\ln F}{dx} = 0 \\ \frac{\partial u}{\partial t} + (u - a)\frac{\partial u}{\partial x} - \frac{1}{\rho a}\left\{\frac{\partial p}{\partial t} + (u - a)\frac{\partial p}{\partial x}\right\} - ua\frac{d\ln F}{dx} = 0\end{gathered} \tag{1.2}$$

Equations (1.2) are satisfied in subregions of parameter continuity. At strong discontinuities the stream parameters satisfy the laws of conservation which in the case of a compression shock are in the same approximation of the form

$$[\rho(u - n)] = 0, \quad [p + \rho(u - n)^2] = 0, \quad [2i + (u - n)^2] = 0 \tag{1.3}$$

where brackets denote the remainder of the combination of parameters related to opposite sides of the shock, which appear within these, n is the shock velocity. If $x = x_s(t)$ is the shock equation of motion, then

$$n = x_s^\cdot(t) \tag{1.4}$$

where the dot denotes a total derivative with respect to time.

The equations and formulas (1.1) – (1.4) together with initial and boundary conditions which are formulated below completely determine the evolution of flow in the channel. Initial conditions must specify the parameter distribution throughout the channel at $t=0$. The number and form of boundary conditions at channel inlet and outlet sections depend on the relation between u and a at these sections. Thus, for example, if the velocity at inlet is supersonic $(u > a)$, all gas parameters must be specified as functions of time. If these parameters are constant, we consider the flow at channel inlet as unperturbed.

2. In investigations of a stationary flow stability we use the equations and formulas at the shock, which are derived from those presented above by the linearization of the latter. This implies the smallness of nonstationary perturbations of parameters. If the stationary flow contains a discontinuity surface (closing shock) whose position is made to coincide with the reference point for x, the latter assumption means that $x_s(t)$ and $n(t)$ are small in comparison with the channel length and the velocity of gas in it.

In linearizing any of the parameters, for instance velocity, we represent it in the form $u(x, t) = U(x)\{1 + u_n(x, t)\}$, where the capital letter denotes the corresponding stationary parameter and $u_n(x, t)$ is the relative nonstationary deviation. An exception is the velocity of shock (in stationary conditions the shock is also stationary) for which the notation $U_- n$, where U_- is the stationary velocity to the left of the closing shock (at $x = 0$), is used. (Gas flows from left to right). Distribution of "large" (stationary) parameters satisfies equations

$$\frac{dU}{dx} = \frac{U}{M^2 - 1}\frac{d\ln F}{dx}, \quad \frac{dP}{dx} = \frac{\mathrm{R}U^2}{1 - M^2}\frac{d\ln F}{dx} \tag{2.1}$$
$$\frac{d\mathrm{R}}{dx} = \frac{\mathrm{R}M^2}{1 - M^2}\frac{d\ln F}{dx} \quad \left(M = \frac{U}{A}\right)$$

which are obtained from (1.2) by omitting in the latter derivatives with respect to time, and in which P, R and A are the stationary pressure, density and the speed of sound, respectively.

Perturbations of parameters at the shock surface become discontinuous and satisfy there the relationships implied by (1.3). In linearizing these relationships it is convenient to have them defined not at the moving discontinuity, i.e. not at $x = x_s(t)$, but at its stationary position at $(x = 0)$. The linearization and transfer to cross section $x = 0$ is carried out with the use of formulas for derivatives of stationary parameters in (2.1) and of the additional assumption of smallness of derivatives of nonstationary additions with respect to x. The latter implies that the linear analysis is restricted to cases in which the contribution of high-frequency components to spectral expansions $u_n(x, t), \ldots$ is not great. It should be noted that in the case of high-frequency perturbations other nonlinear effects become important [12].

If at the channel inlet the flow remains unperturbed during a fairly long interval of time and continues to be unperturbed thereafter, as assumed in the case considered below, then for $t \geqslant 0$ there are no perturbations anywhere to the left of the shock. For these conditions the linearized formulas derived from (1.3) and (1.4) assume (for a perfect gas) the form

$$u_{n+} + \rho_{n+} + (1 - K)n = 0, \quad 2u_{n+} + (1 / \varkappa M_+^2)p_{n+} + \rho_{n+} + \tag{2.2}$$
$$(K - 1)\Delta \ln F = 0$$
$$p_{n+} - N\rho_{n+} + E\Delta \ln F_s = 0, \qquad n = x_s^{\boldsymbol\cdot} / U_-$$
$$\Delta \ln F_s \equiv \ln F(x_s) - \ln F(0) = Yx_s$$
$$\left(K = \frac{U_-}{U_+}, \quad N = \frac{\varkappa\{2 + (\varkappa - 1)M_-^2\}M_-^2}{1 - \varkappa + 2\varkappa M_-^2}\right.$$
$$\left.E = \frac{(\varkappa - N)(M_+^2 - M_-^2)}{(1 - M_+^2)(1 - M_-^2)}, \quad Y = \left(\frac{d\ln F}{dx}\right)_{x=0}\right)$$

Here and in what follows lower case letter denote relative deviations of parameters from their stationary values (for the same x) and the subscripts minus and plus denote parameters upstream and downstream of the closing shock at $x = 0$, respectively.

The linearization of system (1.2) and all subsequent investigations are carried out on the additional assumption of the quasi-cylindricity of the channel cross section to the right of the closing shock. This assumption means that for $x > 0$ any stationary parameter, for instance velocity, can be presented in the form $U(x) = U_+ \{1 + u_e(x)\}$, with $|u_e(x)| \ll 1$ and $|du_e(x)/dx| \ll 1$. Substituting similar expressions into (2.1), discarding terms of second and higher order, and integrating the derived linear system, we obtain

$$u_e(x) = \Delta \ln F(x) / (M_+^2 - 1), \quad p_e(x) = \varkappa M_+^2 \Delta \ln F(x) / (1 - M_+^2), \quad \rho_e(x) = M_+^2 \Delta \ln F(x) / (1 - M_+^2) \tag{2.3}$$
$$(\Delta \ln F(x) \equiv \ln F(x) - \ln F(0))$$

in whose derivation it was taken into account that in accordance with the definition of quantities with subscript e $u_e(0) = p_e(0) = \rho_e(0) = 0$.

In the case of a quasi-cylindrical channel we have within the accuracy of smalls of higher order $u(x, t) = U_+ \{1 + u_e(x) + u_n(x, t)\}$. Substituting this and similar expressions for $p(x, t)$ and $\rho(x, t)$ into (1.2), carrying out linearization and taking into account that $u_e(x)$, $p_e(x)$ and $\rho_e(x)$ are derived by formulas (2.3), we obtain for the nonstationary additions the following system:

$$\left(\frac{\partial}{\partial t} + U_+ \frac{\partial}{\partial x}\right)(p_n - \varkappa \rho_n) = 0, \quad \left\{\frac{\partial}{\partial t} + (U_+ + A_+)\frac{\partial}{\partial x}\right\}\left(u_n + \frac{1}{\varkappa M_+} p_n\right) = 0$$
$$\left\{\frac{\partial}{\partial t} + (U_+ - A_+)\frac{\partial}{\partial x}\right\}\left(u_n - \frac{1}{\varkappa M_+} p_n\right) = 0$$

whose solution is of the form

$$p_n(x, t) - \varkappa \rho_n(x, t) = S(\zeta) \tag{2.4}$$
$$u_n(x, t) + (1/\varkappa M_+) p_n(x, t) = 2R(\xi)$$
$$u_n(x, t) - (1/\varkappa M_+) p_n(x, t) = 2L(\eta)$$
$$(\zeta = x - U_+ t, \quad \xi = x - (U_+ + A_+)t, \quad \eta = x - (U_+ - A_+)t)$$

where the factor 2 in the right-hand parts of the second and third equalities is introduced for the sake of simplifying subsequent formulas, and S, R and L are arbitrary functions. Function S is constant along every particle trajectory, while R and L are constant along the trajectories of acoustic waves propagating over particles in the direction of flow and against it, respectively. In the considered approximation these functions represent the entropy of the right- and left-hand Riemann invariants. We call these the S-, R- and L-waves.

Solving (2.4) for u_n, p_n and ρ_n, we obtain

$$u_n(x, t) = R(\xi) + L(\eta), \quad p_n(x, t) = \{R(\xi) - L(\eta)\}\varkappa M_+ \tag{2.5}$$
$$\rho_n(x, t) = \{R(\xi) - L(\eta)\} M_+ - S(\zeta)\varkappa^-$$

At the shock stationary position ($x = 0$) functions R, L and S must satisfy the linear relationships obtained by the substitution of (2.5) into (2.2). Since only the perturbations determined by the left-hand invariant reach the shock from the right, it is expedient to rewrite the corresponding relationships in the form of formulas defining the reflected invariants (R_+ and S_+) and the shock velocity n in terms of L_+ and of the shift of the shock from its stationary position or of increments $\Delta \ln F_s$ corresponding to such shift. These formulas are of the form

$$R_+ = \varphi L_+ - \psi \Delta \ln F_s, \qquad S_+ = \varphi' L_+ - \psi' \Delta \ln F_s \qquad (2.6)$$
$$n = \mu L_+ - \beta \Delta \ln F_s$$

The derived formulas with appropriately specified coefficients $\varphi, \psi, \ldots$ are valid for a fairly wide class of strong discontinuities (e.g., detonation waves and condensation shocks considered in a single-front approximation). In the case of a compression shock these coefficients are functions of M_- and $\varkappa$ only, and in conformity with (2.2) and (2.4)

$$\varphi = \frac{(1 - 2M_+) N + \varkappa M_+^2}{(1 + 2M_+) N + \varkappa M_+^2}, \quad \psi = \frac{(K-1) N + E}{(1 + 2M_+) N + \varkappa M_+^2} M_+ \qquad (2.7)$$
$$\varphi' = M_+ (1 - \varphi)(\varkappa - N)\varkappa / N, \quad \psi' = \{E - M_+ (\varkappa - N)\psi\}\varkappa / N$$
$$\mu = \frac{1 - M_+ - (1 + M_+)\varphi}{(K-1) M_+}, \quad \beta = 1 - \frac{(1 + M_+)\psi}{(K-1) M_+}$$

If at the channel outlet the velocity is subsonic, the R- and S-waves reach it from the channel side and may be reflected in the form of L-waves. Taking this into consideration we define the boundary conditions at the channel outlet by

$$L = \chi R + \chi' S \quad \text{for } x = 1 \qquad (2.8)$$

where χ and χ' are reflection coefficients which unlike the coefficients $\varphi, \psi, \ldots$ in (2.6) depend not only on stream parameters at the indicated cross section, but also on the equipment adjacent to the investigated channel from the right. In deriving (2.8) l° was taken as the distance from the stationary shock to the channel outlet. For u_*° and F_*° we select the stationary flow velocity downstream of the shock and the cross-sectional area of the channel at $x = 0$, respectively. For such u_*° and F_*°

$$U_- = K, \quad \Delta \ln F = \ln F, \quad \zeta = x - t \qquad (2.9)$$
$$\xi = x - \left(1 + \frac{1}{M_+}\right) t, \quad \eta = x - \left(1 - \frac{1}{M_+}\right) t$$

3. In the investigation of flow evolution in a channel an important part is played by the times τ_s, τ_r and τ_l, which are needed for waves of related kind to pass from the closing shock to the channel outlet. In dimensionless form $\tau_s = 1$, $\tau_r = M_+ / (1 + M_+)$ and $\tau_l = M_+ / (1 - M_+)$. The time it takes an acoustic wave originating at the cross section $x = 0$ in the form of an R-wave and reflected at cross section $x = 1$ in the form of an L-wave, to cover the whole of that path is $\tau = \tau_r + \tau_l = 2M_+ / (1 - M_+^2)$.

Let us examine the behavior of the flow produced as the result of perturbing a certain stationary flow mode which corresponds to specified supersonic parameters at the channel inlet and a given pressure p_b at its outlet ($x = 1$). The closing shock (or discontinuity of some other kind) in stationary mode is located at $x = 0$ and $M_+ < 1$, i.e. the flow in region $0 < x < 1$ is subsonic. The stationary state of the stream can be disturbed by perturbations emanating from the channel inlet and outlet, as well as by any arbitrary external factors (force, thermal effects, etc.). All such factors are assumed to be absent at the initial instant of time $t_0 > \tau$ (in particular, the flow upstream of the shock displaced from its stationary position is unperturbed), and the reflection condition (2.8) is satisfied at cross section $x = 1$ at least during a time interval not shorter than τ. In the first stage of our investigation we set in that condition $\chi \neq 0$ and $\chi' = 0$, which corresponds to the reflection of R-waves only. Let us examine the behavior of flow for a fairly long t on the assumption that the above conditions remain valid for $t > t_0$).

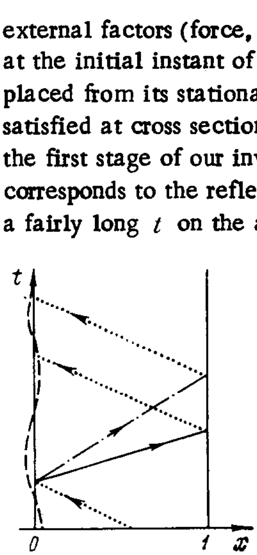

Fig. 1

Note that the analysis of behavior of $x_s(t)$ and $L_+(t)$ provides a complete picture of the behavior of flow. The first two equalities of (2.6) with allowance for (2.9) and for $\ln F_s = Yx_s(t)$ make it possible to determine $R_+(t)$ and $S_+(t)$. The latter prove to be linear combinations of $x_s(t)$ and $L_+(t)$ and owing to their properties are maintained along lines $\xi = \mathrm{const}$ and $\zeta = \mathrm{const}$, respectively.

Functions $x_s(t)$ and $L_+(t)$ satisfy a system of two differential-difference equations, the first of which is derived from the last relationship in (2.6). After elimination of n with allowance for (2.2) and (2.9) it assumes the form

$$x_s^{\bullet}(t) = \mu_0 L_+(t) - \beta_0 Y x_s(t) \quad (\mu_0 = K\mu, \ \beta_0 = K\beta) \tag{3.1}$$

To obtain the second equation we consider the pattern of reflection of R- and L-waves from the shock and from the channel outlet. The L-wave which at instant $t \geqslant t_0$ reached the cross section $x = 0$ is generated by the reflection of an R-wave at instant $t - \tau_l$ from cross section $x = 1$. The latter had left cross section $x = 0$ at instant $t - \tau_l - \tau_r = t - \tau$. Trajectories of the R-, L- and S-waves in the xt-plane are shown in Fig. 1 by solid, dot and dash-dot lines, respectively; arrows indicate the direction of wave propagation and the dash line represents the trajectory of the closing shock. Note that in the considered linear approximation the reflection of L-waves from the shock is replaced by their reflection from cross section $x = 0$. The only perturbation reaching the shock (the upper one at $x = 0$) at instant $t - \tau$, is the L-wave. Hence for $\chi' = 0$ from (2.6) and (2.8) we obtain

$$L_+(t) = \varphi_0 L_+(t - \tau) - \psi_0 Y x_s(t - \tau) \quad (\varphi_0 = \chi\varphi, \ \psi_0 = \chi\psi) \tag{3.2}$$

where φ_0 and ψ_0 are to be taken as the combined coefficients of reflection from the shock and from the channel outlet. It is convenient to use in the analysis besides (3.2) the equation

$$L_+^{\bullet}(t) + \psi_0 Y x_s^{\bullet}(t - \tau) - \varphi_0 L_+^{\bullet}(t - \tau) = 0 \tag{3.3}$$

which is obtained by differentiating (3.2) with respect to t.

Equations (3.1) and (3.3) constitute a system of two differential-difference equations of the neutral kind [15] whose solution for $t \geqslant t_0$ is completely determined by the "input data" along segment $t_0 - \tau \leqslant t \leqslant t_0$. These data specify in that segment two functions $x_s(t) = x_{s0}(t)$ and $L_+(t) = L_{0+}(t)$. In the considered case it is sufficient to specify $x_s(t)$ only at one point (e.g. at $t = t_0$), since $x_{s0}(t)$ is uniquely determined in that segment by (3.1) when function $L_{0+}(t)$ is known for $t_0 - \tau \leqslant t \leqslant t_0$.

For known initial conditions the behavior of the solution of Eqs. (3.1) and (3.3) is determined by the disposition in the complex plane of the roots of that system. The disposition is defined by [15]

$$\lambda\ [(\lambda + \beta_0 Y)\ (1 - \varphi_0 e^{-\tau\lambda}) + \mu_0\psi_0 Y e^{-\tau\lambda}] = 0 \qquad (3.4)$$

For all solutions of (3.1) and (3.3) to be bounded for $t \to \infty$ (i.e. for the initial stationary flow to be stable) it is sufficient according to [15] to have the following conditions satisfied: first, the real parts of all roots of (3.4) must not be positive and, second, every root of (3.4) with a zero real part must be simple. The first condition is also the necessary condition.

The root $\lambda = 0$ is simple. It is convenient to write the equation which determines the infinite sequence of remaining roots of (3.4) as

$$pe^z + q - ze^z + \varphi_0 z = 0 \qquad (3.5)$$
$$(p = -\tau\beta_0 Y, \quad q = \tau Y\ (\beta_0\varphi_0 - \mu_0\psi_0), \quad z = \tau\lambda)$$

The analysis of this equation may be carried out by a method similar to that used in [15] for the case of $\varphi_0 = 0$. Omitting the details of the analysis, we present only the final results. The necessary and sufficient conditions for all roots of (3.5) to have negative real parts require that the four inequalities

$$|\varphi_0| < 1, \quad p < 1 - \varphi_0, \quad p < -q < \sqrt{\alpha^2(1 - \varphi^2{}_0) + p^2} \qquad (3.6)$$

be simultaneously satisfied. In these inequalities α is to be taken as the root of equation $\sin\alpha\ /\ (\cos\alpha - \varphi_0) = \alpha\ /\ p$ when $p \neq 0$. If $p = 0$, it is necessary to set in (3.6) $\alpha = \arccos\varphi_0$. In both cases α must be taken from the interval $0 < \alpha < \pi$.

The first of conditions (3.6) has a particularly simple meaning, namely, that the combined coefficient of reflection of an acoustic wave from the discontinuity and from the channel outlet must be smaller than unity.

An important aspect should be noted. In the absence of entropy wave reflection ($\chi' = 0$) the linearized input problem contains seven parameters, viz. coefficients φ, ψ, μ and β, which appear in conditions (2.6) at the discontinuity, parameter Y which defines the shape of the channel at cross section $x = 0$, the lag time τ and the coefficient χ of the R-wave reflection from the channel outlet. As implied by (3.6), the behavior of flow is determined not by individual parameters, but by three combinations of these: $\varphi_0 = \chi\varphi$, $p = -p_0 Y$ and $q = -q_0\chi Y$, where $p_0 = \tau\beta_0$ and $q_0 = \tau\ (\mu_0\psi - \beta_0\varphi)$, as well as by φ which is a function of $\varkappa$, by the parameters of gas upstream of the discontinuity (in the case of a compression shock also by the Mach number M_-) and by the kind of the discontinuity. For a flow specified upstream of the discontinuity and given parameters Y and χ it is always possible to verify if conditions (3.6) – which guarantee the stability of the considered flow – are satisfied. If an exact equality is obtained instead of any of the inequalities in (3.6), Eq. (3.4) has roots with zero real parts, generally different from $\lambda = 0$. Supplementary investigations are then required for determining the behavior of solution. A reversal of the inequality sign of even a single inequality in (3.6) indicated the instability of the initial stationary flow.

The above shows that conditions (3.6) make it possible to determine regions of stable and unstable flows in the governing parameters of the problem. To illustrate the results of such determination the case of the closing compression shock in a perfect gas with $\varkappa = 1.4$ is shown in Fig. 2. The curves are plotted in the χW_--plane, where W_- is the ratio of flow velocity upstream of the shock to the critical velocity (in stationary conditions). To facilitate the description of these results we denote by (3.6.1),....., (3.6.4) the inequalities in (3.6) after the equality sign has been substituted in these for the inequality sign.

Let the shock be localized in the divergent part of the channel $(Y > 0)$. The lower boundary of stable flows is then defined by the solid line curve in Fig. 2. Its shape and position is independent of parameter Y. The equation of this curve $\chi = -p_0 / q_0$ is readily derived from (3.6.3). Curves which form the upper boundary of stable flows depend on Y. These curves are shown in Fig. 2 for $Y = 0, 0.1, 0.2, 0.3, 0.4$, and ∞ by dash lines (numbers indicate values of Y for each curve). For $Y = 0$ the upper boundary is defined by the equation $\chi = 1 / \varphi$ which for this particular value of Y can be derived either from (3.6.1) or (3.6.4). Boundaries corresponding to $Y \neq 0$ are determined by Eq. (3.6.4); with increasing Y they are monotonically displaced downward. For this reason the limit curve corresponding to $Y = \infty$ is also shown in Fig. 2.

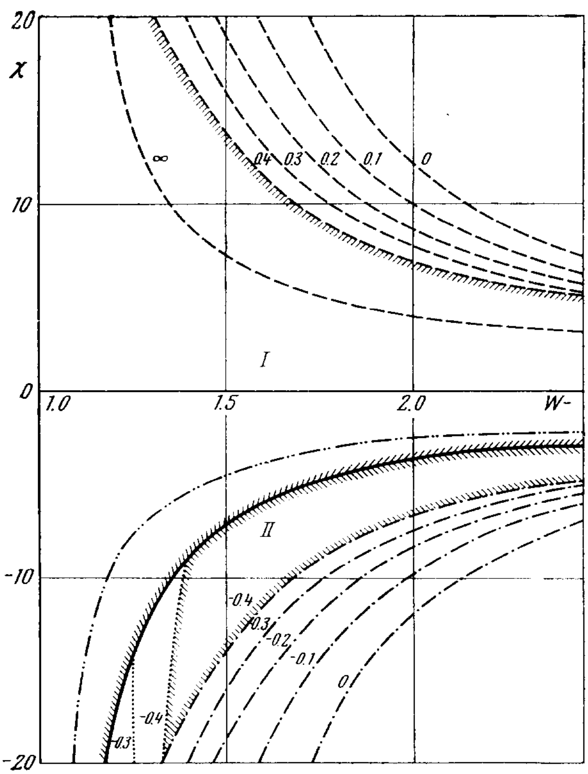

Fig. 2

It is defined by the equation $\chi = p_0 / q_0$. Since the quasi-cylindrical approximation is valid only for fairly small Y, hence in an arbitrary case in which it is still valid the upper boundary of the stable flow region lies above that curve.

In the case of a convergent channel $(Y < 0)$ the region of stable flows lies above the solid line curve, as opposed to the case of $Y > 0$. The stability region is bounded on the left partly by the solid line and partly by the curve $\chi = (1 + p_0 Y) / \varphi$, whose equation is obtained from (3.6.2). Within the limits of variation of χ the corresponding section of the boundary appears in Fig. 2 only for fairly large $|Y|$. For $Y = -0.3$ and -0.4 these segments are shown in that figure by dotted lines. Lower boundaries of the stability region depend for $Y < 0$ on the value of Y, and are shown by dash-dot lines for five of its values. For $Y = 0$ the boundary is defined by the equation $\chi = -1 / \varphi$ which is a corollary of (3.6.1), as well as of (3.6.4). For $Y < 0$ the lower boundaries are determined by Eq. (3.6.4) which implies their displacement upward and, for $Y = -\infty$, merging with the solid line curve. In fact, the stable flow region vanishes much earlier, owing to the displacement of the left-hand boundary (shown by the dotted line) to the right and downward. To make the above clear boundaries of regions I and II – which are regions of stability for $Y = 0.4$ and $Y = -0.4$, respectively, – are shaded in Fig. 2 from inside.

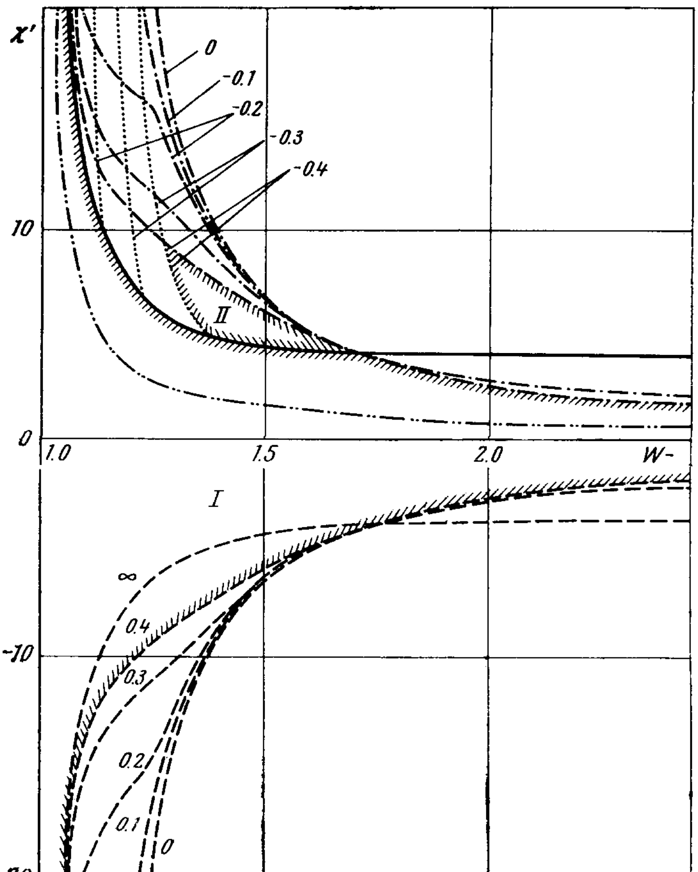

Fig. 2

The above analysis is readily applicable to the case in which only entropy waves, i.e. when in (2.8) $\chi = 0$ and $\chi' \neq 0$ are reflected from the channel outlet. All conditions derived above remain valid, if χ', φ', ψ' and $\tau' = 1 / (1 - M_+)$, where φ' and ψ' obtained for the closing shock from (2.7) are substituted for χ, φ, ψ, and τ. The results of related investigations plotted in the $\chi' W_-$-plane are shown in Fig. 3, where the various curves (shown by solid, dash, etc. lines) and also regions I and II have the same meaning as in Fig. 2. The essential difference between these figures is that curves lying in Fig. 2 above (below) the axis of abscissas appear in Fig. 3 below (above) that axis. Moreover, in the case of entropy wave reflection the curves intersect each other, which does not occur in Fig. 2.

4. Conditions (3.5) and for $\varkappa = 1.4$ Figs. 2 and 3 make it possible to solve the problem of stationary flow stability in a quasi-cylindrical channel with a closing compression shock for a wide range of boundary conditions.

As an example, let us consider several specific methods of formulating such conditions.

The axis of abscissas lying entirely in region I in Figs. 2 and 3 correspond to a complete absence of perturbation reflection from the channel outlet ($\chi = \chi' = 0$ for all W_-). Hence in this case the flow with a closing shock is stable in a divergent channel and unstable in a convergent one. As previously indicated, this conclusion was arrived at by Chernyi in 1953. It can be directly derived from (2.6), (2.8) and (2.2) or from Eqs. (3.1) and (3.3) which in that case are of the form

$$x_s^{\cdot}(t) = -\beta_0 Y x_s(t), \quad L_+(t) = 0$$

From this with allowance for (2.8) and $\chi = \chi' = 0$ we obtain

$$x_s(t) = x_s(t_0) \exp\{\beta_0 Y (t_0 - t)\}, \;\; L_+(t) \equiv 0 \qquad (4.1)$$

Computations of β_0 by formulas (2.7) show that it is positive for all W_- and $\varkappa$. Hence, in accordance with (4.1) the flow is stable for $Y > 0$ and unstable for $Y < 0$. This conclusion is also valid for a number of other boundary conditions at cross section $x = 1$. Thus, when pressure is fixed at the channel outlet, then in accordance with (2.5) and (2.8) $\chi = 1$ and $\chi' = 0$ for all W_-. As seen from Fig. 2, the straight line $\chi = 1$ lies entirely in region I.

When the Mach number or the gas flow rate are fixed at the channel outlet, the reflection coefficients in (2.8) are defined as follows: $\chi = [(\varkappa - 1) M_+ - 2] / [(\varkappa - 1) M_+ + 2]$, $\chi' = 1 / \varkappa [(\varkappa - 1) M_+ + 2]$, and $\chi = (M_+ + 1) / (M_+ - 1)$ and $\chi' = 1 / \varkappa (1 - M_+)$, respectively. In the first case both reflection coefficients are in the entire range of W_- smaller in modulo than unity. Consequently, curves $\chi = \chi(W_-)$ and $\chi' = \chi'(W_-)$ lie in region I in Figs. 2 and 3. Similar behavior of the condition of flow rate constancy is shown in these figures by dash-double-dot line. Although the related curves lie closer to the boundary of region I than in previous cases, they do not leave it (including $W_- \to 1$ when $\chi \to -\infty$ and $\chi' \to \infty$). Note that all of the above statements are valid for all $\varkappa$ in the range $1.1 - 5/3$.

It should be stressed that a complete investigation of stability in the last two cases requires the taking into account of simultaneous, not separate reflection of R- and S-waves. It can be shown that the latter requirement reduces to the analysis of roots of the characteristic equation

$$pe^z + q - z(e^z - \varphi'_0 e^{kz}) + \varphi_0 z + (q'\tau / \tau') e^{kz} = 0 \;\; (k = (\tau - \tau') / \tau) \qquad (4.2)$$

where φ_0' and q' are derived from φ_0 and q after the substitution of χ', φ', ψ' and τ' for χ, φ, ψ and τ.

The characteristic equation (4.2) is equivalent to a system of three differential-difference equations with two lag (times) τ and τ', and is more complex then, for instance, the characteristic equation of a similar kind obtained in [3], where the left-hand part of the equations is a quasi-polynomial without its principal term. The latter indicates the instability of solution for any other values of all other constants appearing in the equation [15]. Incidentally, the latter aspect was not noted in [3], and this led to the erroneous conclusion about the existence of stable modes of detonation combustion in the theory of the two-front model suggested in that paper. The conclusion about the absence of stable modes of detonation combustion was arrived at in [16] with the use of a somewhat different two-front model, and for the model used in [3] it was obtained in [17]. In [16], as well as here for (3.5), the necessary and sufficient condition for the real parts of roots of the related characteristic equation to be nonnegative is derived. Formulation of similar conditions in the case of (4.2) is difficult. It is nevertheless possible to carry out the necessary analysis for fixed W_-, Y and $\varkappa$ by numerical methods with the use of the "D-subdivision" [18, 19].

Without going into details of such analysis — which could be the subject of a special investigation — we would point out the following. As seen from Figs. 2 and 3, the region of stable modes in the $\chi\chi'$-plane occupies for $Y > 0$ and any fixed W_- some neighborhood of the coordinate origin ($\chi = \chi' = 0$). An idea of the dimensions of that region along the axes χ and χ' of the $\chi\chi'$-plane is provided by the boundary curves plotted in Figs. 2 and 3, with the related points tending to infinity for $W_- \to 1$. As previously noted, the coeffcients χ and χ' do not exceed in modulo unity, if the Mach number is fixed at the channel outlet. Comparison of their values with the boundary values taken from Figs. 2 and 3 shows that in the indicated cases one can hardly expect instability (stability) of flow with a closing shock in a divergent (convergent) channel. Higher values of related coeffcients for a fixed gas flow rate do not permit to make a similar statement with the same degree of confidence in the last case.

Let us conclude with a few general remarks. First, the limitedness of the used here quasi-cylindricity approximation must be stressed. Hence it cannot be excluded that in the case of channels which do not satisfy the condition of quasi-cylindricity the conclusions arrived at above will not hold. This relates in particular to the possible instability of flow with a closing shock in a divergent channel. The second remark relates to the limitedness of the linear approximation in cases in which results of its application indicate an instability of the flow, which is particularly important for transonic velocities.

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